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Nernst Theorem and Entropy of the Axisymmetric Einstein–Maxwell–Dilaton-Axion Black Hole

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The free energy and the entropy of scalar field are calculated by brick-wall in the axisymmetric Einstein–Maxwell–Dilaton-axion black hole. It is shown that when the black hole has inner and outer horizons, the entropy is not only related to the area of an outer horizon but also is the function of the area of an inner horizon. When the area of an inner horizon approaches zero, we can obtain the known conclusion. The entropy expressed by location parameter of outer horizon and inner horizons approaches absolute zero. It obeys Nernst theorem. It can be taken as Planck absolute entropy of a black hole.

Since Bekenstein and Hawking have proposed that the entropy of a black hole is proportional to the area of the horizon (Bekenstein, 1973; Gibbons and Hawking, 1977; Hawking, 1975), the statistical source of the black hole entropy have been studied. Each method of studying the entropy has been given (Cai *et al.*, 1998; Cognola and Lecca, 1998; Hochberg *et al.*, 1993; Lee *et al.*, 1996; Padmanaban, 1989; 't Hooft, 1985). G.'t Hooft's brick-wall method is often used ('t Hooft, 1985). The statistical property of a free scalar field in different kinds of black hole is studied by this method (Shen and Chen, 1999; Solodukhin, 1995a,b). The entropy expressed by the area of horizon has been obtained and the fact that entropy is proportional to the area of the outer horizon has been proved. In Schwarzschild space–time ('t Hooft, 1985), when the cutoff satisfies the proper condition, the entropy has the form of $S = A_H/4$. When the cutoff approaches zero, the entropy diverges. G.'t Hooft thought that this divergence was caused by the infinite density of states near horizon. Afterward the entropy of every black hole is studied by this way (Shen and Chen, 1999; Solodukhin, 1995b); when the brick-wall method is used, we only considered the contribution from the outer horizon and neglected the contribution from the inner horizon. By this way, we obtain that the entropy is proportional to the area of the outer horizon. However, the conclusion does not satisfy Nernst theorem.

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In this paper, in the process of integration, we not only consider the contribution from outer horizon but also consider the effect from inner horizon, and obtain the free energy and entropy of a black hole by the brick-wall method. When the area of a inner horizon approaches zero, the entropy returns to the known result. Further, when the radiation temperature of a black hole approaches absolute zero, the entropy expressed by location parameter of outer and inner horizon approaches zero. It satisfies Nernst theorem.

In curved space–time, the dynamic behavior of the massless particle can be described by Klein–Gordon equation:

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial}{\partial x^{\mu}}\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\right)\psi = 0. \tag{1}$$

The linear element is given in axisymmetric Einstein–Maxwell–Dilaton-axion space–time by Garcia *et al.* (1995)

$$\begin{aligned} ds^2 = &-\frac{\Sigma - a^2\sin^2\theta}{\Delta}\,dt^2 + \frac{\Delta}{\Sigma}\,dr^2 + \Delta\,d\theta^2 \\ &+ \frac{\sin^2\theta}{\Delta}[(r^2 + a^2 - 2Dr)^2 - \Sigma a^2\,\sin^2\theta]\,d\varphi^2 \\ &- \frac{2a\,\sin^2\theta}{\Delta}[(r^2 + a^2 - 2Dr) - \Sigma]\,dt\,d\varphi, \end{aligned} \tag{2}$$

where

$$\Sigma = r^2 - 2mr + a^2, \qquad \Delta = r^2 - 2Dr + a^2\,\cos^2\theta$$

and

$$\begin{aligned} \exp(2\phi) &= \frac{W}{\Delta} = \frac{\omega}{\Delta}(r^2 + a^2\,\cos^2\theta), \quad \omega = \exp(2\phi_0), \\ K_{\alpha} &= K_0 + \frac{2aD\,\cos\theta}{W}, \\ A_t &= \frac{1}{\Delta}(Qr - ga\,\cos\theta), \qquad A_r = A_{\theta} = 0, \\ A_{\varphi} &= \frac{1}{a\Delta}[-Qra^2\,\sin^2\theta + g(r^2 + a^2)\,a\,\cos\theta], \\ M &= m - D, \quad J = a(m - D), \quad Q = \sqrt{2\omega D(D - m)}, \quad \text{and} \quad P = g, \end{aligned}$$

where M, J, Q, and P are mass, angular momentum, charge, and magnetic charge, respectively. We use G.'t Hooft's theory and let

$$\psi(r) = 0 \quad \text{when } r \leq r_+ + \epsilon$$

and

$$\psi(r) = 0 \quad \text{when } r \geq L,$$

where

$$r_+ = \left(M - \frac{Q^2}{2\omega M}\right) + \sqrt{\left(M - \frac{Q^2}{2\omega M}\right)^2 - a^2}$$

is the position of outer horizon of a black hole, ϵ and L are the ultraviolet and infrared cutoffs, respectively, and $L \gg r_+$. Using the WKB approximation, the wave function of Eq. (1) is

$$\psi = \exp[-iEt + im\varphi + iS(r, \theta)] \tag{3}$$

Letting $P_r = \partial_r S$ and $P_\theta = \partial_\theta S$ from (1) and (2) we obtain (Mann *et al.*, 1992):

$$P_r^2 = \left[-g^{tt}E^2 + 2g^{t\varphi}EJ_m - g^{\varphi\varphi}J_m^2 - g^{\theta\theta}P_\theta^2\right]/g^{rr}.$$

The number of micro states with energy E and angular momentum J_m is (Padmanaban, 1989):

$$\Gamma(E, J_m) = \frac{1}{\pi}\int d\varphi\, d\theta \int_{r_+ + \epsilon}^{L} dr$$
$$\times \int dP_\theta \left[\frac{1}{g^{rr}}\left(-g^{tt}E^2 + 2g^{t\varphi}EJ_m - g^{\varphi\varphi}J_m^2 - g^{\theta\theta}P_\theta^2\right)\right]^{1/2}. \tag{4}$$

The free energy of system can be expressed by (Shen and Chen, 1999):

$$F = -\frac{1}{6\pi^2}\int d\varphi\, d\theta \int_{r_+ + \epsilon}^{L} dr \int_0^\infty \frac{E^3\, dE}{e^{\beta E} - 1}\frac{\sqrt{-g}}{(-g'_{tt})^2}, \tag{5}$$

where

$$g'_{tt} = g_{tt} + 2\Omega_0 g_{t\varphi} + \Omega_0^2 g_{\varphi\varphi}$$
$$= \frac{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}{g_{\varphi\varphi}}\left[1 + (\Omega - \Omega_0)^2 \frac{g_{\varphi\varphi}^2}{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}\right], \tag{6}$$

and dragging angular velocity is

$$\Omega = -\frac{d\varphi}{dt} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}}, \tag{7}$$

we can assume that the scalar field rotates with a dragging velocity $\Omega = \Omega_0$ (Shen and Chen, 1999) and obtain

$$\begin{aligned} g'_{tt} &= \frac{g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2}{g_{\varphi\varphi}} \\ &= -\frac{(r-r_+)(r-r_-)(r^2-2Dr+a^2\cos^2\theta)}{(r^2+a^2-2Dr)^2-(r-r_+)(r-r_-)a^2\sin^2\theta}, \end{aligned} \tag{8}$$

where

$$r_- = \left(M - \frac{Q^2}{2\omega M}\right) - \sqrt{\left(M - \frac{Q^2}{2\omega M}\right)^2 - a^2}$$

is the position of inner horizon of a black hole. Substituting (2) and (8) into (5), we have

$$\begin{aligned} F &= -\frac{1}{6\pi^2}\int d\varphi\, d\theta \int_{r_++\epsilon}^{L} dr \int_0^\infty \frac{E^3\, dE}{e^{\beta E}-1} \\ &\quad \times \frac{[(r^2+a^2-2Dr)^2-(r-r_+)(r-r_-)a^2\sin^2\theta]^2\sin\theta}{(r-r_+)^2(r-r_-)^2(r^2-2Dr+a^2\cos^2\theta)} \\ &= -\frac{2\pi}{6\pi^2}\frac{1}{\beta^4}6\times\frac{\pi^4}{90}\int d\theta \int_{r_++\epsilon}^{L}\left[\frac{(r^2+a^2-2Dr)^4}{(r-r_+)^2(r-r_-)^2(r^2-2Dr+a^2\cos^2\theta)}\right. \\ &\quad - \frac{2(r^2+a^2-2Dr)^2a^2\sin^2\theta}{(r-r_+)(r-r_-)(r^2-2Dr+a^2\cos^2\theta)} \\ &\quad \left. + \frac{a^4\sin^4\theta}{(r^2-2Dr+a^2\cos^2\theta)}\right]\sin\theta\, dr. \end{aligned} \tag{9}$$

Because our integral region is out of a black hole horizon, we have $r > a$, then

$$\begin{aligned} F &\approx -\frac{\pi^3}{45\beta^4}2\int_{r_++\epsilon}^{L}\int_0^1 \frac{(r^2+a^2-2Dr)^4\, d[\cos\theta]}{(r-r_+)^2(r-r_-)^2(r^2-2Dr+a^2\cos^2\theta)}\, dr \\ &= -\frac{2\pi^3}{45\beta^4}\int_{r_++\epsilon}^{L}\frac{(r^2+a^2-2Dr)^4}{(r-r_+)^2(r-r_-)^2}\frac{1}{a\sqrt{r^2-2Dr}}\ \text{arc}\, tg\frac{a}{\sqrt{r^2-2Dr}}\, dr \\ &\approx -\frac{2\pi^3}{45\beta^4}\int_{r_++\epsilon}^{L}\frac{(r^2+a^2-2Dr)^4}{(r-r_+)^2(r-r_-)^2}\frac{dr}{(r^2-2Dr)} \end{aligned} \tag{10}$$

In Eq. (10), we only take the divergent term, when $\epsilon \to 0$, and have

$$
\begin{aligned}
&\int_{r_+ + \epsilon}^{L} \frac{(r^2 + a^2 - 2Dr)^4}{(r - r_+)^2 (r - r_-)^2 (r^2 - 2Dr)}\, dr \\
&= \frac{r_+^3 (r_+ + r_- - 2D)^4}{(r_+ - r_-)^2 (r_+ - 2D)\epsilon} + \frac{r_-^3 (r_+ + r_- - 2D)^4}{(r_- - r_+)^2 (r_- - 2D)(r_+ + \epsilon - r_-)} \\
&\quad + \left[4 \frac{r_+^2 (r_+ + r_- - 2D)^3}{(r_+ - r_-)(r_+ - 2D)} + \frac{3 r_+^2 (r_+ + r_- - 2D)^4}{(r_+ - r_-)^2 (r_+ - 2D)} \right. \\
&\quad \left. - \frac{r_+^3 (3r_+ - r_- - 2D)(r_+ + r_- - 2D)}{(r_+ - r_-)^3 (r_+ - 2D)^2} \right] \ln \frac{1}{\epsilon} \\
&\quad + \left[4 \frac{r_-^2 (2r_+ + r_- - 2D)^3}{(r_- - r_+)(r_- - 2D)} + \frac{3 r_-^2 (2r_+ + r_- - 2D)^4}{(r_- - r_+)^2 (r_- - 2D)} \right. \\
&\quad \left. - \frac{r_-^3 (3r_- - r_+ - 2D)(r_+ + r_- - 2D)^4}{(r_- - r_+)^3 (r_- - 2D)^2} \right] \ln \frac{1}{r_+ + \epsilon - r_-}.
\end{aligned}
\tag{11}
$$

Other integral terms which are the contribution from the vacuum surrounding the system at large distances and of little relevance here, and can be neglected. In the approximation, the free energy of a scalar field in the axisymmetric Einstein–Maxwell–Dilaton-axion black hole is

$$
\begin{aligned}
F = -\frac{2\pi^3}{45\beta^4} &\left[\frac{r_+^3 (r_+ + r_- - 2D)^4}{(r_+ - r_-)^2 (r_+ - 2D)\epsilon} + \frac{r_-^3 (r_+ + r_- - 2D)^4}{(r_- - r_+)^2 (r_- - 2D)(r_+ + \epsilon - r_-)} \right. \\
&+ \left(4 \frac{r_+^2 (r_+ + r_- - 2D)^3}{(r_+ - r_-)(r_+ - 2D)} + \frac{3 r_+^2 (r_+ + r_- - 2D)^4}{(r_+ - r_-)^2 (r_+ - 2D)} \right. \\
&- \left. \frac{r_+^3 (3r_+ - r_- - 2D)(r_+ + r_- - 2D)}{(r_+ - r_-)^3 (r_+ - 2D)^2} \right) \ln \frac{1}{\epsilon} \\
&+ \left(4 \frac{r_-^2 (2r_+ + r_- - 2D)^3}{(r_- - r_+)(r_- - 2D)} + \frac{3 r_-^2 (2r_+ + r_- - 2D)^4}{(r_- - r_+)^2 (r_- - 2D)} \right. \\
&- \left. \left. \frac{r_-^3 (3r_- - r_+ - 2D)(r_+ + r_- - 2D)^4}{(r_- - r_+)^3 (r_- - 2D)^2} \right) \ln \frac{1}{r_+ + \epsilon - r_-} \right].
\end{aligned}
\tag{12}
$$

Using the relation between the entropy and free energy of the system

$$
S = \beta^2 \frac{\partial F}{\partial \beta},
\tag{13}
$$

we obtain:

$$S = S_+ + S_-, \tag{14}$$

where

$$S_+ = \frac{8\pi^3}{45\beta^3}\left[\frac{r_+^3(r_+ + r_- - 2D)^4}{(r_+ - r_-)^2(r_+ - 2D)\epsilon} + \left(\frac{4r_+^2(r_+ + r_- - 2D)^3}{(r_+ - r_-)(r_+ - 2D)}\right.\right.$$
$$\left.\left. + \frac{3r_+^2(r_+ + r_- - 2D)^4}{(r_+ - r_-)^2(r_+ - 2D)} - \frac{r_+^3(3r_+ - r_- - 2D)(r_+ + r_- - 2D)^4}{(r_+ - r_-)^3(r_+ - 2D)^2}\right)\ln\frac{1}{\epsilon}\right], \tag{15}$$

$$S_- = \frac{8\pi^3}{45\beta^3}\left[\frac{r_-^3(r_+ + r_- - 2D)^4}{(r_- - r_+)^2(r_- - 2D)}\frac{1}{r_+ + \epsilon - r_-}\right.$$
$$+\left(\frac{4r_-^2(r_+ + r_- - 2D)^3}{(r_- - r_+)(r_- - 2D)} + \frac{3r_-^2(r_+ + r_- - 2D)^4}{(r_- - r_+)^2(r_- - 2D)}\right.$$
$$\left.\left. - \frac{r_-^3(3r_- - r_+ - 2D)(r_+ + r_- - 2D)^4}{(r_- - r_+)^3(r_- - 2D)^2}\right)\ln\frac{1}{r_+ + \epsilon - r_-}\right]. \tag{16}$$

From Eq. (16), when we fix the lower limit of the integral $r_+ + \epsilon$ and exchange r_+ for r_-, we have the same result of (15). So we can take S_+ as a contribution from the outer horizon and take S_- as a contribution from the inner horizon.

Substituting

$$\beta = \frac{1}{T_H} = \frac{2\pi}{\kappa} = \frac{4\pi(r_+^2 + a^2 - 2Dr_+)}{r_+ - r_-}$$

into (15) and (16), we get

$$S_+ = \frac{\pi}{90\beta}\left[\frac{r_+(r_+ + r_- - 2D)^2}{(r_+ - 2D)\epsilon} + \left(\frac{4(r_+ + r_- - 2D)(r_+ - r_-)}{r_+ - 2D}\right.\right.$$
$$\left.\left.\times\frac{3(r_+ + r_- - 2D)^2}{r_+ - 2D} - \frac{r_+(3r_+ - r_- - 2D)(r_+ + r_- - 2D)^2}{(r_+ - r_-)(r_+ - 2D)^2}\right)\ln\frac{1}{\epsilon}\right] \tag{17}$$

$$S_- = \frac{\pi}{90\beta}\left[\frac{r_-^3(r_+ + r_- - 2D)^2}{r_+^2(r_- - 2D)}\frac{1}{r_+ + \epsilon - r_-}\right.$$
$$+ \left(\frac{4r_-^2(r_+ + r_- - 2D)(r_+ - r_-)}{(r_- - 2D)r_+^2} + \frac{3r_-^2(r_+ + r_- - 2D)^2}{(r_- - 2D)r_+^2}\right.$$
$$\left.\left. - \frac{r_-^3(3r_- - r_+ - 2D)(r_+ + r_- - 2D)^2}{(r_- - r_+)r_+^2(r_- - 2D)^2}\right)\ln\frac{1}{r_+ + \epsilon - r_-}\right]. \qquad (18)$$

Using the relation between the area and location of horizon

$$A_H = 4\pi(r_+^2 + a^2 - 2DR_+), \qquad (19)$$

we know that (17) and (18) can be written as

$$S_+ = \frac{(r_+ + r_- - 2D)}{90\beta\epsilon(r_+ - 2D)}\frac{A_H}{4} + \frac{1}{90\beta(r_+ - 2D)r_+}\frac{A_H}{4}$$
$$\times\left[7r_+ - r_- - 6D - \frac{r_+(3r_+ - r_- - 2D)(r_+ + r_- - 2D)}{(r_+ - r_-)(r_+ - 2D)}\right]\ln\frac{1}{\epsilon}, \qquad (20)$$

$$S_- = \frac{r_-^3(r_+ + r_- - 2D)}{90\beta r_+^3(r_- - 2D)}\frac{A_H}{4}\frac{1}{r_+ + \epsilon - r_-} + \frac{\pi}{90\beta(r_- - 2D)}\frac{A_H}{4r_+^3}$$
$$\times\left[7r_-^3 - r_+r_-^2 - 6Dr_-^2 - \frac{r_-^3(3r_- - r_+ - 2D)(r_+ + r_- - 2D)}{(r_- - r_+)(r_- - 2D)}\right]$$
$$\times\ln\frac{1}{r_+ + \epsilon - r_-}. \qquad (21)$$

When $r_+ \to r_-$,

$$S_+ = \frac{2r_+ - 2D}{90\beta\epsilon(r_+ - 2D)}\frac{A_H}{4} - \frac{2r_+ - 2D}{90(r_+ - 2D)^2}\frac{A_H}{16\pi r_+}\ln\frac{1}{\epsilon}, \qquad (22)$$

$$S_- = \frac{2r_+ - 2D}{90\beta\epsilon(r_+ - 2D)}\frac{A_H}{4} + \frac{2r_+ - 2D}{90(r_+ - 2D)^2}\frac{A_H}{16\pi r_+}\ln\frac{1}{\epsilon}, \qquad (23)$$

$$S = S_+ + S_- = 0 \qquad (24)$$

When $r_+ \to r_-$, temperature and entropy of the black hole approach zero. It obeys the Nernst theorem.

When $r_- \to 0$,

$$S_+ = \frac{1}{90\beta\epsilon}\frac{A_H}{4} + \frac{A_H(r_+ - D)}{90\beta(r_+ - 2D)r_+}\ln\frac{1}{\epsilon}\cdot \qquad (25)$$

It returns to the result of ’t Hooft (1985). From (20) and (21), when $r_+ \gg r_-$, the contribution to entropy from r_- can be neglected. From the first term of (20) and (21), in other case, the contribution from r_- cannot be neglected.

On the basis of this analysis, starting with the scalar field equation of free particles, we obtain the wave function by WKB approximation. The free energy and entropy of a scalar field are calculated by brick-wall method. It is obtained that the entropy of axisymmetric Einstein–Maxwell–Dilation-axion black hole is the function of outer and inner horizons. When $r_+ \gg r_-$ or r_- is very small comparing with r_+, the contribution to entropy from inner horizon can be neglected. When r_+ and r_- are at the same quantitative standing, the contribution to entropy from inner horizon cannot be neglected. If we take limit, the conclusion returns to the known result (Shen and Chen, 1999). Errors caused by defining the entropy of a black hole only by outer horizon surface are corrected. The entropy defined now obeys the Nernst theorem.

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